ANSWER KEY: NORMAL DISTRIBUTION
This answer key provides solutions to the corresponding student activity sheet.

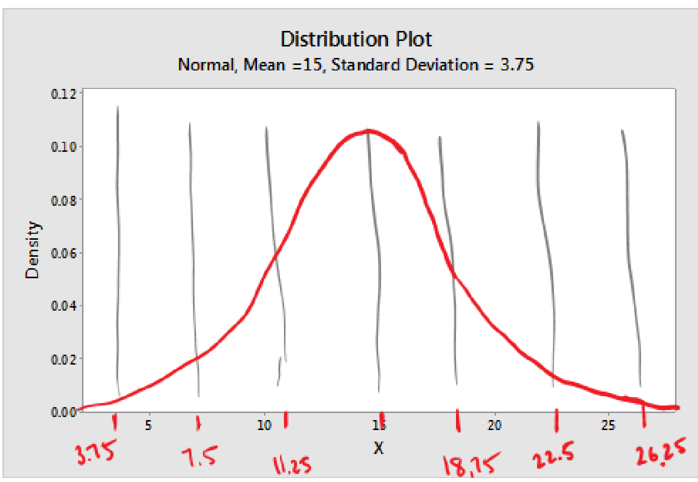
Normal Distribution

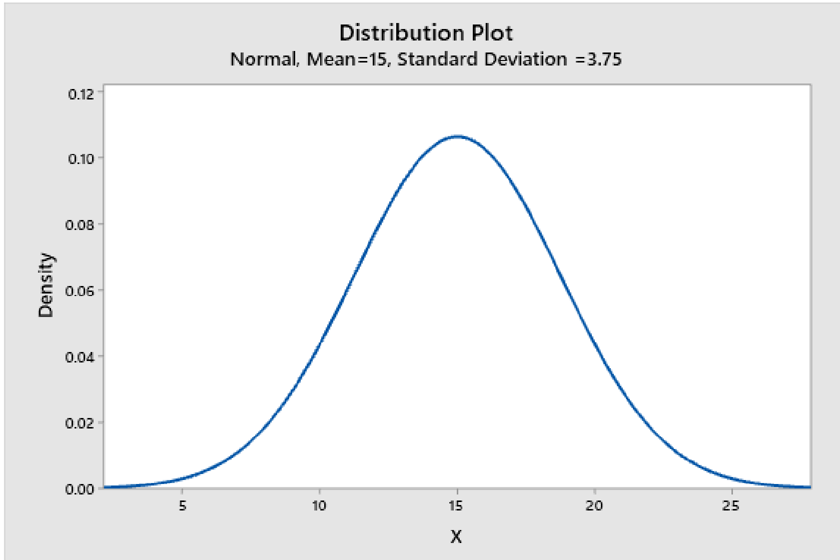
A dataset is not provided for these exercises.

Exercise 1

(a) Sketch (as best you can) the normal curve with μ = 15 seconds and σ = 3.75 seconds.

Solution: The peak of the curve is at μ = 15 with approximately 68% of the data within 1 standard deviation (between 11.25 and 18.75), approximately 95% of the data within 2 standard deviations (between 7.5 and 22.5), and approximately 99.7% within 3 standard deviations (between 3.75 and 26.25). Look for the graph to have the bell shape and be close to the graph drawn below. Note that concavity switches directions at 1σ in both directions from the mean.





(b) Take at most 10 test “flights” at https://www.echalk.co.uk/amusements/Games/pilotTrainer/pilotTrainer.html and record your survival times. Of the flights, what is your longest survival time (in seconds)?

Solution: Answers will vary! My best time was 13.14.

(c) What proportion of survival times, according to the normal distribution with μ = 15 seconds and σ = 3.75 seconds, is longer than your best survival time? Hint: Convert your best survival time to a *z*-score and use the normal table to calculate the answer.

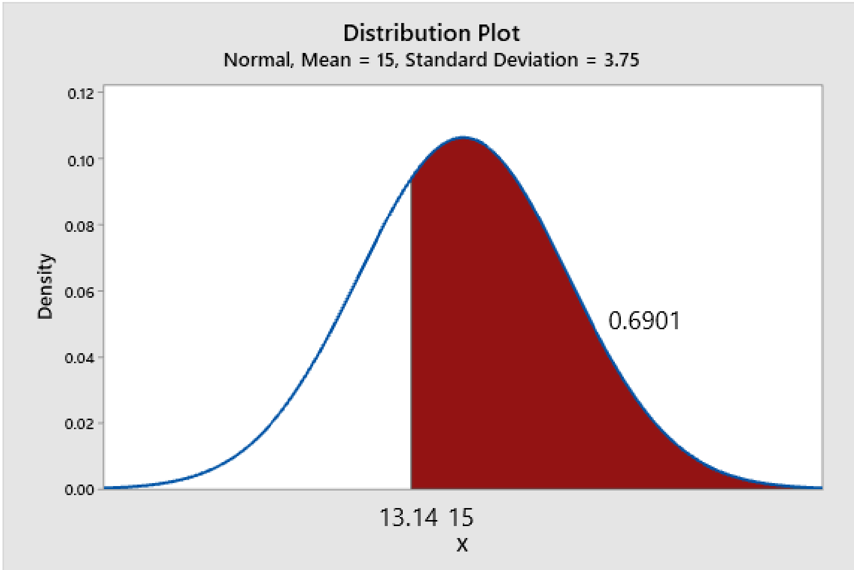
Solution: The proportion will depend on the student’s best survival time. If the survival time is greater than *x* = 15, then the proportion will be less than 0.5. If the survival time is less than *x* = 15, then the proportion will be greater than 0.5. Using the survival time plot from part (a), the instructor can determine if the student is on the right track with his or her proportion. The proportion of survival times better than my best time of *x* = 13.14 seconds is:

$$P\left(Z > \frac{13.14-15}{3.75}\right) \cong P(Z > -0.50) \cong P(Z > -0.50) \cong \mathbf{0.6915}$$

(d) What proportion of survival times is shorter than your best survival time?

Solution: The answer is the complement to the answer in part **(c)**, namely approximately 1 – 0.6915 = **0.3085**. Make sure the student’s answers for parts **(c)** and **(d)** add to 1.

We can use Minitab to compute the probabilities in parts **(c)** and **(d)**. Note that the Minitab answer is more accurate than the answer obtained from the standard normal table. Here is Minitab’s proportion for part **(c)** using my best survival time of 13.14 seconds.



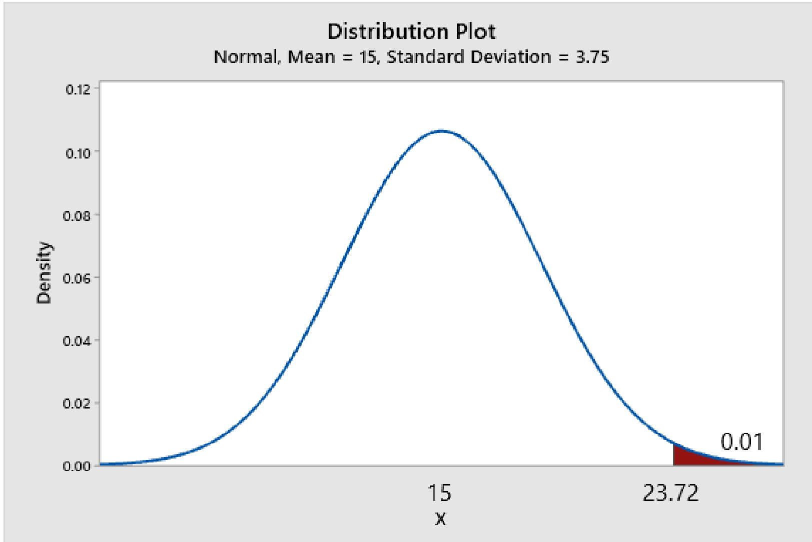
(e) I'm taking applications for my Halo 3 team, but I only want to interview applicants whose flight survival times are in the top 1%. What should I set as the minimum survival time necessary for consideration? First do this by-hand using the standard normal table. Hint: Determine the *z*-score corresponding to the top 0.01 area in the right tail of the standard normal curve. Then back transform the *z*-score into the original *x* scale, flight time in seconds.

Solution: Since we are determining a right tail proportion, we need the *z*-score in the normal table corresponding to 1 – 0.01 = 0.99. Looking in the body of the normal table for the value closest to 0.99 gives us ***z* = 2.33**. Now we need to "unstandardize" the *z*-score back to the original *x* (survival time) scale. We know that *x* must satisfy the following equation:

$$\frac{x - 15}{3.75} = 2.33$$

Solving this equation for *x* gives us a survival time of 15 + 2.33 * 3.75 = **23.74 seconds**. Based on my best time of 13.14 seconds, I could not be a part of my own Halo 3 team!

We can verify this answer using Minitab, which gives a more accurate value of **23.72 seconds**.



(f) Based on your best survival time, could you apply for my Halo 3 team? Why or why not?

Solution: If a student's best survival time is greater than 23.74, then they can be on the Halo 3 Team. Otherwise, the survival time isn't good enough to be on the team.

Exercise 2

According to the collective experience of generations of pediatricians, pregnancy durations, let's call them *X*, tend to be normally distributed with μ = 266 days and σ = 16 days. Perform a probability calculation that addresses San Diego Reader's credibility, presuming she was pregnant for 308 days. What would you conclude and why? Show the work behind your answer, including a *z*-score and probability.

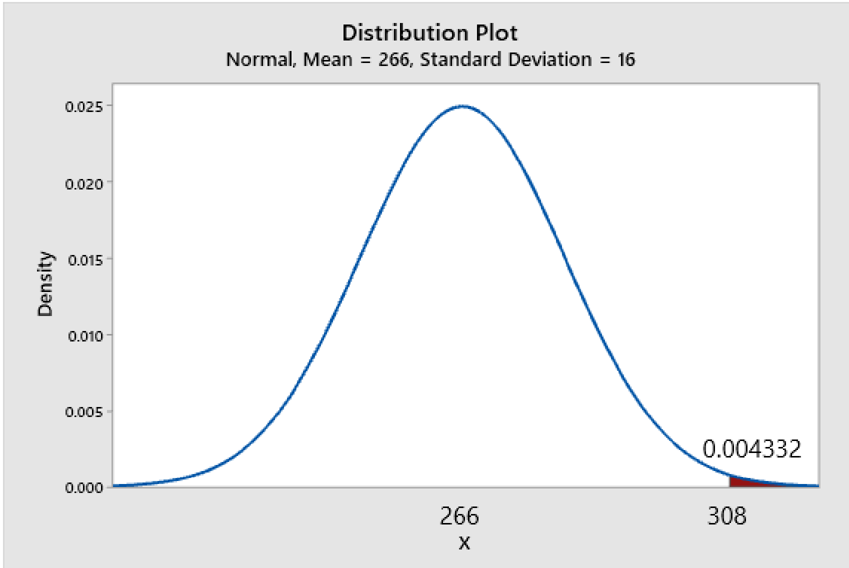
Solution: Let *X* represent pregnancy lengths, where *X* has a normal distribution with mean μ = 266 days and standard deviation σ = 16 days. The *z*-score for 308 days is:

$$z = \frac{308 - 266}{16} \cong 2.63$$

The probability that a pregnancy lasts longer than 308 days is:

$$P(X > 308) \cong P(Z > 2.63) \cong \mathbf{0.00427}$$

The Minitab plots report the probabilities as **0.004332**.



The probability of giving birth after 308 days is 0.0043. Although being 2.63 standard deviations away from the mean is unusual, it's not impossible. In the 1970's, paternity tests were becoming more accurate, and I'd suggest that the San Diego Reader should use one to erase any doubt from her husband's mind (assuming she is telling the truth!). On the other hand, in the United States alone in 1970 there were around 3.7 million births. With 3.7 million births, we would expect approximately 16,000 pregnancies (3.7 million * 0.0043) to last as long as 308 days.

Exercise 3

(a) The Inn receives customer complaints for serving "tiny" filets, where tiny qualifies as a filet weighing less than 6.21 oz. If catfish filets served at the Inn have weights that are normally distributed with mean μ = 6.3 oz. and standard deviation σ = 0.04 oz., determine a tiny filet's *z*-score and the probability that a randomly selected customer will be served one. Construct a Minitab plot showing the desired probability.

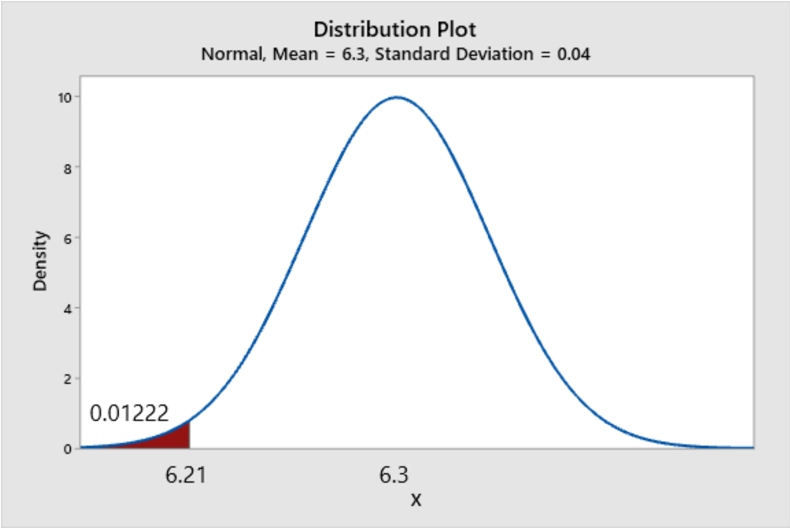
Solution: Let *X* be the weight of the randomly selected catfish filet. The *z*-score of a tiny filet is:

$$z = \frac{6.21 - 6.3}{0.04} = \mathbf{-2.25}$$

The probability that a customer will be served a filet weighing 6.21 oz. or less is:

$$P(X < 6.21) = P(Z < -2.25) \cong \mathbf{0.01222}$$

The Minitab plot also reports the probability as **0.01222**.



(b) A waiter at the Inn has been serving all day and wants to leave for the evening with a good tip. He randomly selects filets in hopes of getting a jumbo one, where jumbo qualifies as a filet weighing more than 6.4 oz. Determine a jumbo filet's *z*-score and the probability that a randomly selected filet will turn out to be jumbo. Construct a Minitab plot showing the desired probability.

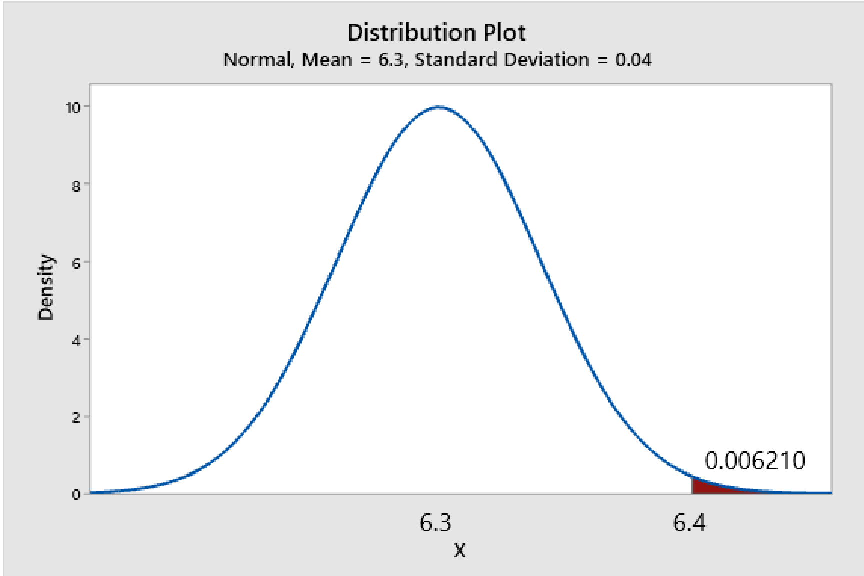
Solution: Let *X* be the weight of the randomly selected catfish filet. The *z*-score of a jumbo filet is:

$$z = \frac{6.4 - 6.3}{0.04} = \mathbf{2.5}$$

The probability that a customer will be served a filet weighing 6.4 oz. or more is:

$$P(X > 6.4) = P(Z > 2.5) \cong \mathbf{0.00621}$$

The Minitab plot reports the probability as **0.006210**.



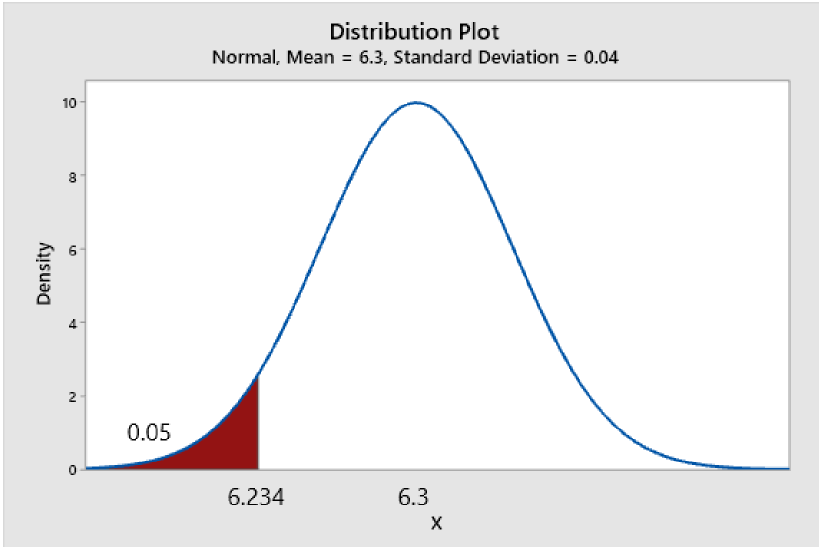
(c) Determine the filet weight at which only 5% of filets weigh less than it. Use the normal table and show your calculations. Construct a Minitab plot showing the desired weight.

Solution: Since we are determining a left tail proportion, we need the *z*-score in the normal table corresponding to 0.05. Looking in the body of the normal table for the value closest to 0.05 gives us ***z* = -1.64**. Now we need to "unstandardize" the *z*-score back to the original *x* (weight) scale. We know that *x* must satisfy the following equation:

$$\frac{x - 6.3}{0.04} = -1.64$$

Solving this equation for *x* gives us a weight of 6.3 – 1.64 * 0.04 = **6.2344 oz**.

We can verify this answer using Minitab, which gives the value **6.234 oz**.



Exercise 4

(a) If *X* represents the duration of an eight-week-old baby's smile, where *X* is normally distributed with mean μ = 9.1 seconds and standard deviation σ= 2.2 seconds, what is the probability that a randomly chosen eight-week-old baby smiles between 6 and 12 seconds? Compute the answer by hand and check it with Minitab using the process described below.

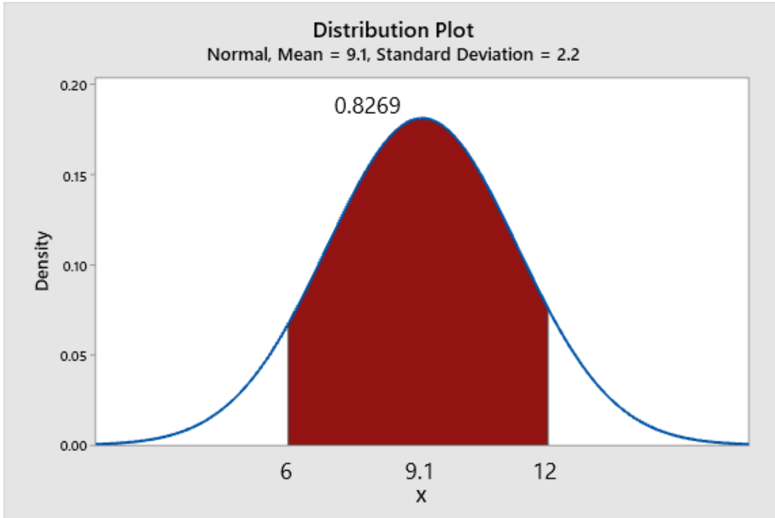
Solution: Since *X* is assumed to have a normal distribution with μ = 9.1 and σ = 2.2, then the *z*-scores of 6 and 12 are:

$$z = \frac{6-9.1}{2.2} \cong -1.41 \text{ and } z = \frac{12-9.1}{2.2} \cong 1.32$$

The probability that the smile duration is between 6 and 12 seconds can be found using the normal table.

$$P(6 < X < 12) = P(X < 12) - P(X < 6) = P(Z < 1.32) - P(Z < -1.41) \cong 0.90658 - 0.07927 = \mathbf{0.82731}$$

We can verify this answer using Minitab, which returns the following graph:



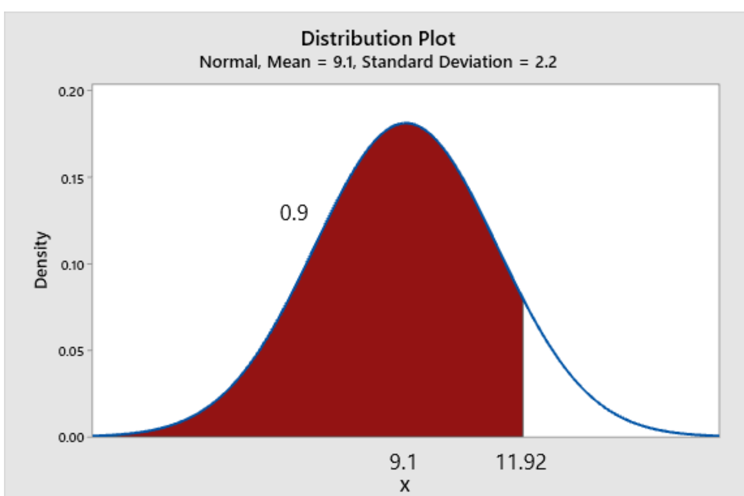
(b) Determine the 90th percentile for the duration of an eight-week-old baby's smile. That is, find the value *x* such that the probability of being less than or equal to *x* is 0.90. Set up the correct expression and show your work. Check your answer with Minitab.

Solution: Since we are determining a left tail proportion, we need the *z*-score in the normal table corresponding to 0.90. Looking in the body of the normal table for the value closest to 0.90 gives us ***z* = 1.28**. Now we need to "unstandardize" the *z*-score back to the original *x* (duration of baby's smile) scale. We know that *x* must satisfy the following equation:

$$\frac{x-9.1}{2.2}=1.28$$

Solving this equation for *x* gives us a time of 9.1 + 1.28 * 2.2 = **11.916 seconds.**

We can verify this answer using Minitab, which gives the value **11.92 seconds**.



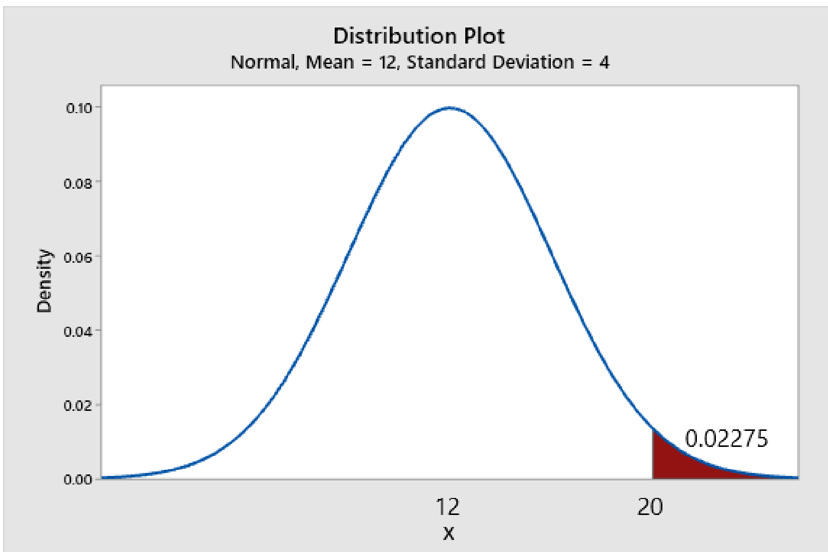
Exercise 5

(a) If the length of a walk with Aiko is normally distributed with a mean of $\mu = 12$ minutes and standard deviation of $\sigma = 4$ minutes, what proportion of walks last longer than 20 minutes? Answer to four decimal places.

Solution: Let X represent the length of walks with Aiko.

$$P(X > 20) = P\left(Z > \frac{20 - 12}{4}\right) = P(Z > 2) \cong \mathbf{0.0228}$$

Using Minitab:



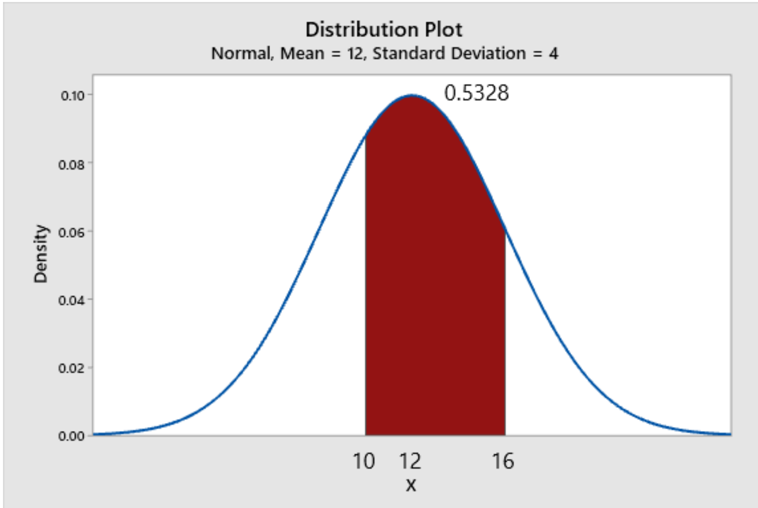
(b) What proportion of walks last between 10 and 16 minutes? Answer to four decimal places.

Solution: Again, let X represent the length of walks with Aiko.

$$P(10 < X < 16) = P(X < 16) - P(X < 10) = P\left(Z < \frac{16 - 12}{4}\right) - P\left(Z < \frac{10 - 12}{4}\right) = P(Z < 1) - P(Z < -0.5)$$

$$\cong 0.84134 - 0.30854 = \mathbf{0.5328}$$

Using Minitab:



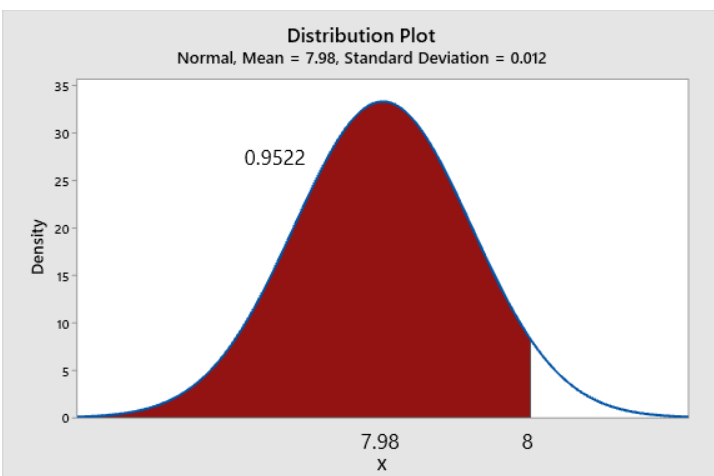
Exercise 6

(a) If the amount of chips put into bags can be adequately modelled with a normal distribution with μ = 7.98 oz. and σ = 0.012 oz., what proportion of bags filled by this machine will contain less than 8 oz. of chips? Show your work.

Solution: Let X be the amount of chips put into a bag. Then the proportion of bags with fewer than 8 oz. of chips is:

$$P(X < 8) = P\left(Z < \frac{8 - 7.98}{0.012}\right) \cong P(Z < 1.67) \cong 0.95254$$

Using Minitab:

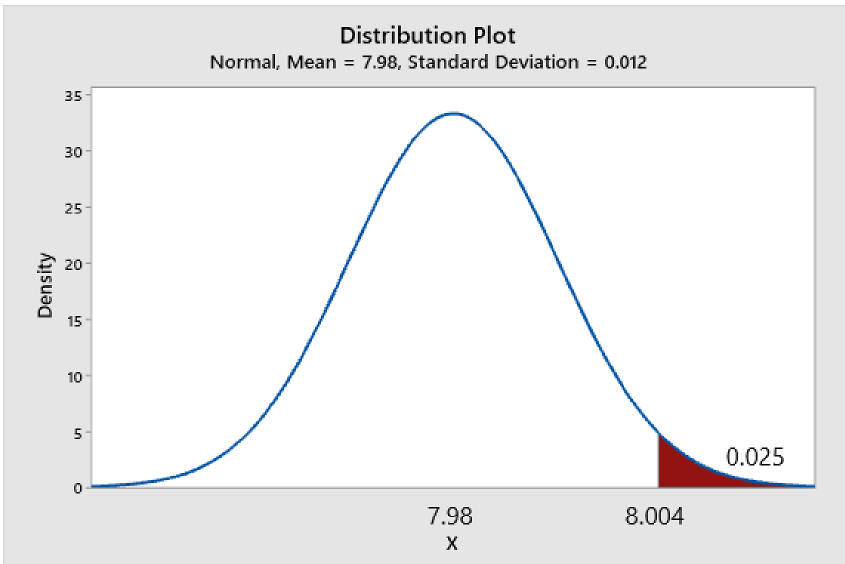


(b) What amount of chips (in ounces) is exceeded only 2.5% of the time? Show your work. Answer to 3 or 4 decimal places.

Solution: Using the *z* table, the area to the right of *z* = 1.96 is approximately 0.025. Thus, the amount of chips exceeded only 2.5% of the time is the solution to the equation below: **8.0035 oz.**

$$\frac{x - 7.98}{0.012} = 1.96$$

Using Minitab:



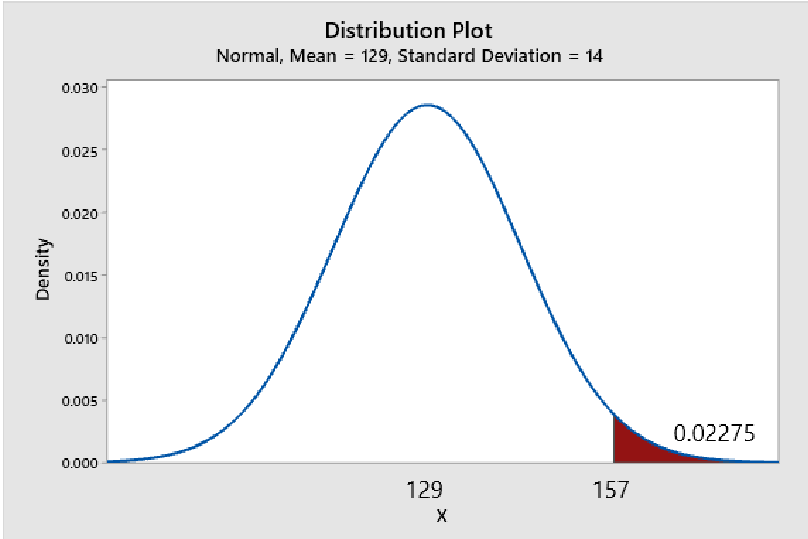
Exercise 7

(a) If ACL reconstruction surgery at high-volume hospitals has a mean time of 129 minutes and a standard deviation of 14 minutes, what is the probability that a randomly selected ACL surgery at a high-volume hospital requires a time that is more than two standard deviations above the mean? Show your work, either by hand or in Minitab.

Solution: Let *X* be the ACL surgery time. We simply need to compute $P(Z > 2)$, which is verified below.

$$P(X > 129 + 2 * 14) = P(X > 157) = P\left(Z > \frac{157 - 129}{14}\right) = P(Z > 2) \cong 0.0228$$

Using Minitab:

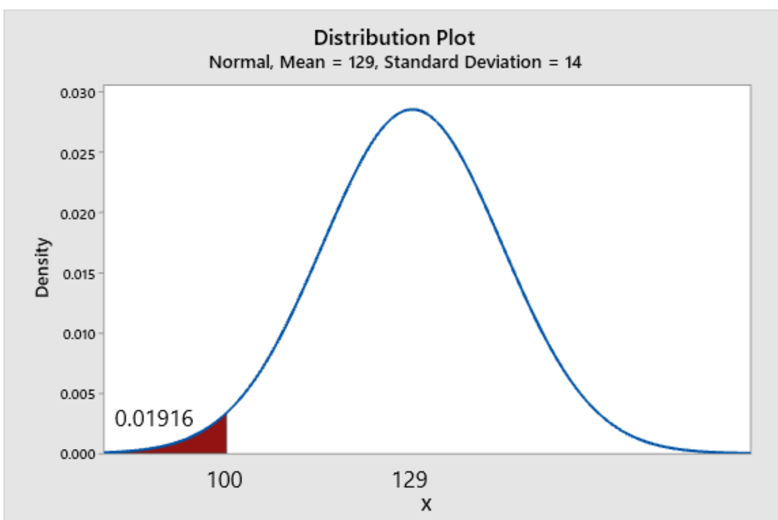


(b) What is the probability that your ACL surgery at a high-volume hospital is completed in less than 100 minutes? Show your work, either by hand or in Minitab.

Solution: Let *X* be the ACL surgery time.

$$P(X < 100) = P\left(Z < \frac{100 - 129}{14}\right) \cong P(Z < -2.07) \cong \mathbf{0.01923}$$

Using Minitab:

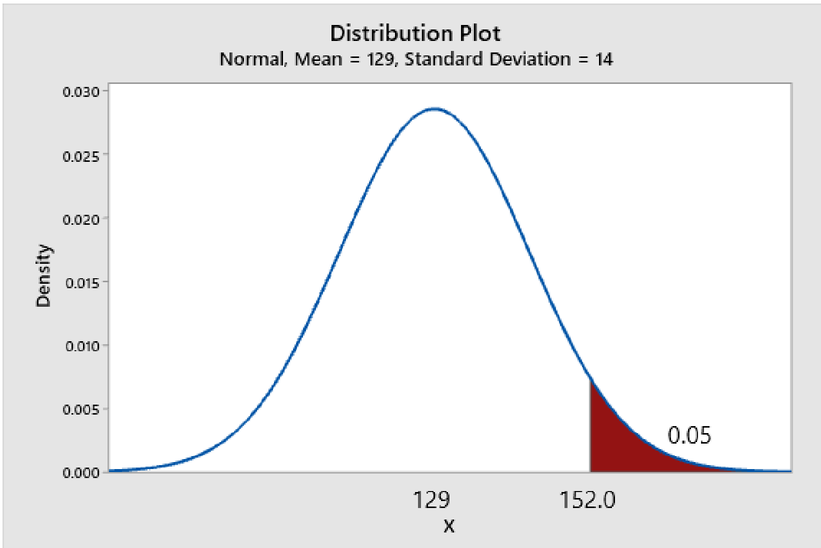


(c) What surgery time (in minutes) is exceeded only 5% of the time? Show your work, either by hand or using Minitab.

Solution: Using the z table, the area to the right of z is approximately 0.05. Thus, the surgery time exceeded only 5% of the time is approximately 151.96 minutes, which is the solution to the equation below.

$$\frac{x - 129}{14} = 1.64$$

Using Minitab:



Exercise 8

The distribution of the 2010 overall MCAT scores is given in the table below. To get into Harvard, you need to have a score above 36. What percentile does this score represent? Note: The p^{th} percentile has p percent of scores less than it. Show your work, either by hand or using Minitab.

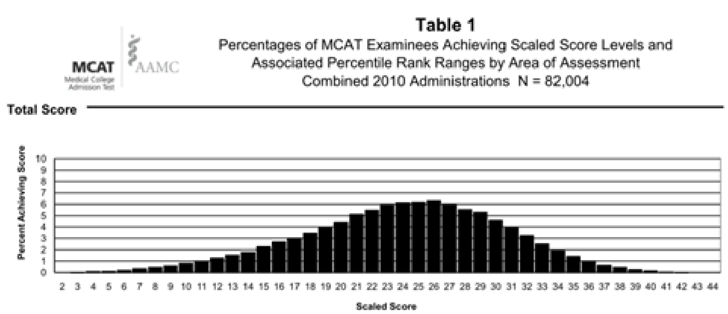


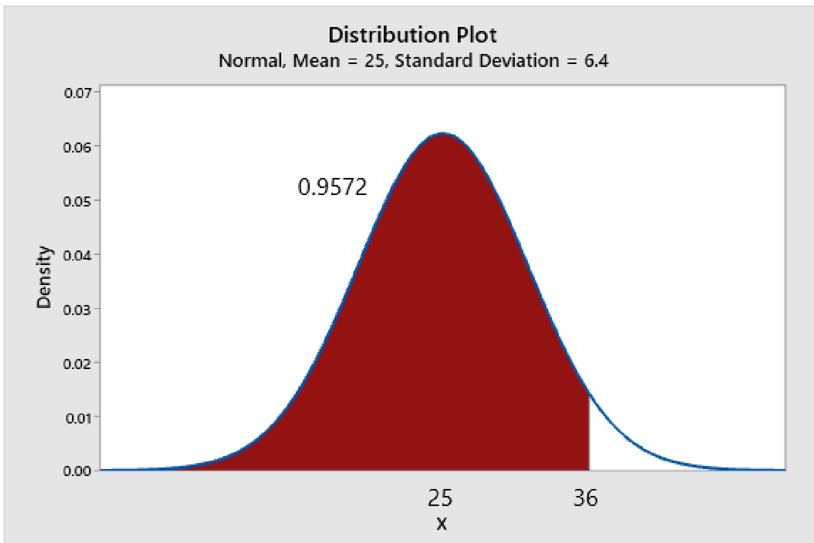
Table Information: Mean = 25, Standard Deviation = 6.4, Maximum = 44

Solution: Let *X* be the MCAT score of a randomly chosen test taker. By considering the shape of the distribution in the table, *X* can be approximated by a normal distribution with mean μ = 25 and standard deviation σ = 6.4. In order to the determine the percentile associated with *x* = 36, we can find its *z*-score and use the standard normal table.

$$P(X < 36) = P\left(Z < \frac{36 - 25}{6.4}\right) \cong \boldsymbol{P(Z < 1.72) \cong 0.95728}$$

Thus, the percentile is the **95.7th percentile or 96th percentile.**

Using Minitab:



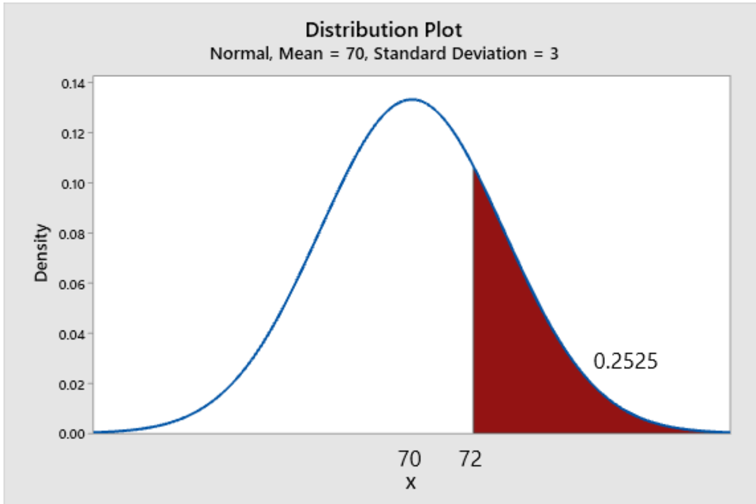
Exercise 9

(a) If a man is selected at random and adult men's heights (in inches) are normally distributed with μ = 70 and σ = 3, what is the probability that he is eligible for the Beanstalk Club?

Solution: Let *X* represent adult men's heights. To be a member of the "Beanstalk Club," a man has to be 72 inches tall. The probability of this occurring is **0.25143**.

$$P(X \geq 72) = P\left(Z \geq \frac{72 - 70}{3}\right) = P\left(Z \geq \frac{2}{3}\right) \cong \mathbf{0.25143}$$

Using Minitab:

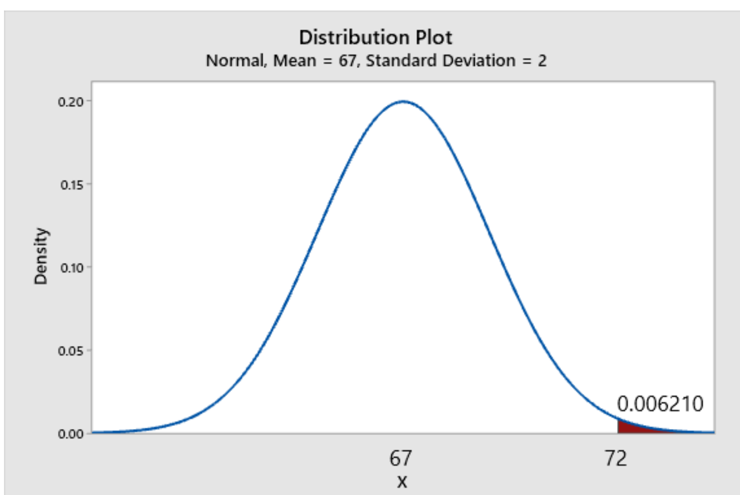


(b) If a woman is selected at random and adult women's heights (in inches) are normally distributed with μ = 67 and σ = 2, what is the probability that she is eligible for the Beanstalk Club?

Solution: Let Y represent adult women's heights. To be a member of the "Beanstalk Club," a woman has to be 72 inches tall. The probability of this occurring is **0.00621**.

$$P(Y \geq 72) = P\left(Z \geq \frac{72-67}{2}\right) = P\left(Z \geq \frac{5}{2}\right) \cong \mathbf{0.00621}$$

Using Minitab:



(c) [BONUS] If a man and a woman are selected at random from their respective populations, what is the probability that the pair is eligible for the Beanstalk Club? Assume that we pick the man and woman independently; for example, the man and woman are not a couple or related to each other. Answer to 4 decimal places.

Solution: Since the man and woman are chosen independently of each other, then $P(X \geq 72 \text{ and } Y \geq 72)$ can be determined by multiplying $P(X \geq 72)$ and $P(Y \geq 72)$. Using the probabilities from the Minitab graphs:

$$P(\text{Both Beanstalk Members}) = P(X \geq 72 \cap Y \geq 72) = P(X \geq 72) * P(Y \geq 72) \cong 0.2525 * 0.0062 \cong \mathbf{0.0016}$$

Exercise 10

(a) If the grades for a certain exam are normally distributed with mean 67 and variance 64, what percent of students get B's (80-90)?

Solution: Let X represent exam grades. Then X is normally distributed with mean μ = 67 and standard deviation σ = 8. [Careful – don't use the variance as the standard deviation!] The proportion of students getting B's is approximately:

$$P(80 < X < 90) = P(X < 90) - P(X < 80) = P\left(Z < \frac{90-67}{8}\right) - P\left(Z < \frac{80-67}{8}\right) \cong P(Z < 2.88) - P(Z < 1.63) \cong 0.99801 - 0.94845 \cong \mathbf{0.0496}.$$

(b) What percent of students get F's (< 60)?

Solution: Let X represent exam grades. The proportion of students getting F's is approximately:

$$P(0 < X < 60) = P(X < 60) - P(X < 0) = P\left(Z < \frac{60-67}{8}\right) - P\left(Z < \frac{0-67}{8}\right) \cong P(Z < -0.88) - P(Z < -8.38) \cong 0.18943 - 0 \cong \mathbf{0.18943}$$

Exercise 11

Manufactured parts have lifetimes in hours, X, that are normally distributed with mean 1000 and standard deviation 100. If $800 \leq X \leq 1200$, the manufacturer makes a profit of \$50 per part. If $X > 1200$, the profit per part is \$75. Otherwise, the manufacturer loses \$25 per part. What is the expected profit per part?

Solution: Using Minitab (see graphs below), the proportion of parts lasting between 800 and 1200 hours is approximately 0.9545. The proportion of parts lasting more than 1200 hours is 0.02275. The proportion of parts lasting less than 800 hours is 0.02275.

Given these proportions and corresponding profits, the expected profit per part is:

$$\$50 * P(800 \leq X \leq 1200) + \$75 * P(X > 1200) - \$25 * P(X < 800) =$$
$$\$50 * 0.9545 + \$75 * 0.02275 - \$25 * 0.02275 \cong \mathbf{\$48.86}$$