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Normal Distribution

Lesson Overview

The **normal distribution** is probably the most widely used of all probability models. Its popularity is partly due to the Central Limit Theorem, which states that as the sample size increases, the sampling distribution of the mean can be approximated by a normal distribution. Some variables typically modeled by the normal distribution include height, weight, IQ, memory, reading ability, etc.

The normal distribution is completely described by two population parameters – the mean μ and the standard deviation σ. The mean of a normal distribution is its center, and the standard deviation measures the spread of the distribution about the mean.

In this lesson, the normal distribution will be defined, and we'll use it to determine probabilities for variables that can be modeled by it. We'll compute these probabilities using a normal table and Minitab.

Prerequisites

None. Although integration of the normal probability density function is included, covering it is optional.

Learning Targets

This lesson teaches students how to:

- View, recognize, and graph a normal distribution with mean μ and standard deviation σ. The **68–95–99.7 rule**, also known as the **3σ rule** or **empirical rule**, will be discussed.
- Determine probabilities associated with a normal distribution using a normal probability table and Minitab.

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- **Standardize** values from any normal distribution. In other words, convert x values associated with a normal distribution with mean μ and standard deviation σ to **z-scores**. A z-score will be defined in the lesson.
- Work backwards when given a probability from a normal distribution to determine its corresponding x value.

Time Required

It will take the instructor 60 minutes in class to introduce the normal distribution and work through examples for determining probabilities and corresponding x values. We recommend starting the activity sheet in class so that students can ask the instructor questions while working on it. The exercises on the activity sheet will take 60 minutes, and they can be used as homework or quiz problems.

Materials Required

- Minitab desktop (20 or higher) or Minitab web app
- Internet access for simulation and data-generating activities.

Assessment

The activity sheet contains exercises for students to assess their understanding of the learning targets for this lesson.

Possible Extensions

The instructor may want to extend this to the **Binomial Distribution** and/or **Other Distributions** after this lesson so that students can have exposure to additional distributions.

References

A simulation of the Quincunx: http://www.mathsisfun.com/data/quincunx.html

Instructor Notes with Examples

The Normal Distribution

The **normal distribution** is one of the most widely used probability models. It is also referred to as the **Gaussian distribution** after its inventor, German mathematician Carl Gauss. The probability density function of a normal distribution is actually on the 10 Deutsche Mark (on right) in honor of Gauss.

Graphing the probability density function – which describes probabilities over intervals – associated with a normal distribution reveals a bell-shaped curve with a center at **mean μ** and a spread of **standard deviation σ**, where σ is always positive. The normal distribution is completely described by these **two population parameters**.

Below are **graphs of normal distributions** with various means and standard deviations. Notice that the bell of the curve is located at the mean μ. The shape of the curve is dependent on the standard deviation σ, where a larger σ corresponds to a wider curve.
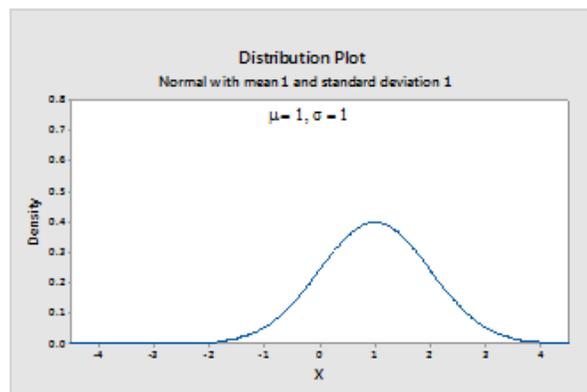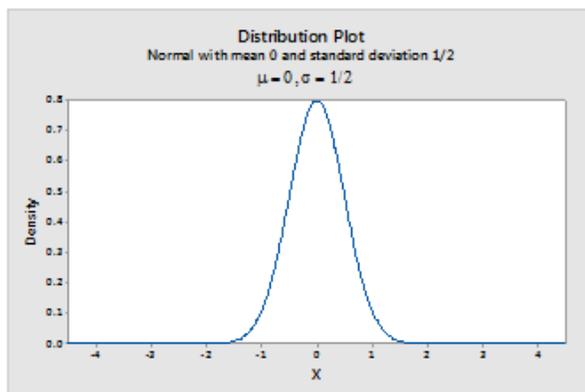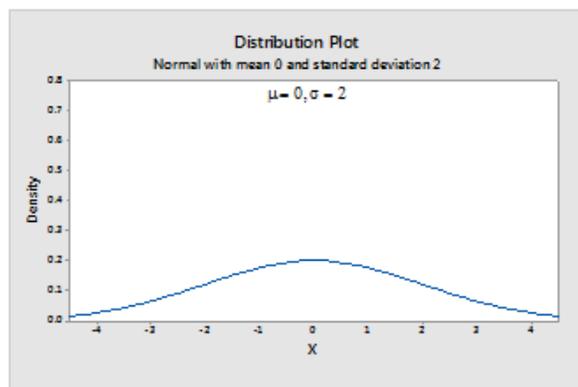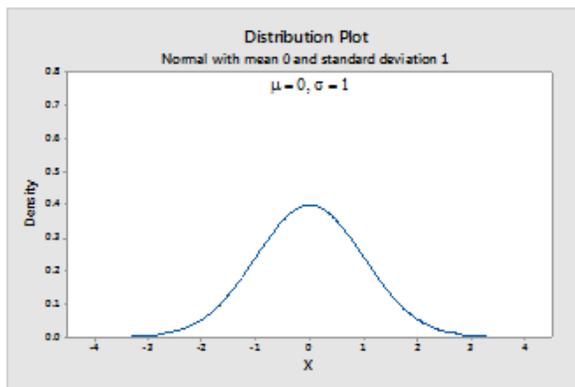
Many variables are adequately modeled by the normal distribution, including height, weight, IQ, errors in measurements, exam scores, blood pressure, etc. Another reason for the popularity of the normal distribution is the **Central Limit Theorem** (CLT).

The Central Limit Theorem

Definition: The **Central Limit Theorem** states that as the sample size increases, the sampling distribution of the mean can be approximated by a normal distribution. In other words, if we repeatedly take independent random samples of size *n* from *any* population (regardless of the population's distribution), then when *n* is large, the distribution of the sample means will approach a normal distribution. Note that the Central Limit Theorem is discussed in greater detail in the ***Sampling Distribution of Xbar*** lesson.

Demonstration: The **Quincunx**, first illustrated by Sir Francis Galton, illustrates a simple process of summing random variables that gives rise to the familiar "bell curve" of the normal distribution. A simulation of the Quincunx is available at the following website:
http://www.mathsisfun.com/data/quincunx.html.

- A Quincunx is a triangular array of pins.
- Balls are dropped onto the top pin and pass through every row of pins until they land in a given position at the bottom. Each row of pins can be thought of as a random event with two choices – "left" or "right."
- Once at the bottom, the balls stack up to show the frequency of balls that have landed at a given position.
- The final position of each ball is determined by the sum of independent, random events each with probability *p* = 0.5 as to whether the ball drops to the left or the right of the pin.

At first there is no pattern, but after many balls have been dropped, the stacks conform to a bell shape. This is because there are more right/left row combinations that lead a ball back to the

Balls are dropped onto the top pin
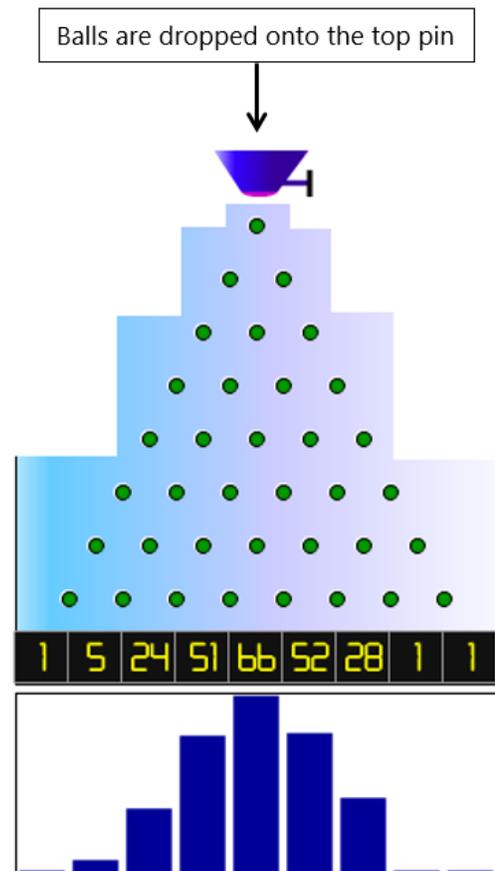
Balls stack at the bottom in their landing positions. More balls fall near the center than at the tails.

center of the distribution rather than the tails. The only way for a ball to reach the far-right tail given *n* Quincunx rows is for the ball to bounce right *n* times, which has probability $(1/2)^n$ given a 50%/50% right/left ball bounce probability.

The Normal Probability Density Function

Optional: For students who have taken calculus, the function that produces the bell shape is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ for all real number } x's$$

The function *f(x)* is called a **probability density function** and graphing it for any μ and positive σ yields a bell curve centered at μ with spread σ. The function *f(x)* is a legitimate probability density function since *f(x)* > 0 for all real numbers *x* and the area under the curve of *f(x)* is 1.

The normal probability density function with:

- Mean μ=0 and standard deviation σ=1 is written as: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ for all real number x's.

- Mean μ=3 and standard deviation σ=2 is written as: $f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-3)^2}{2\cdot 2^2}}$ for all real number x's.

The 68-95-99.7 Rule

Although there are many normal distributions (since there are infinitely many choices for μ and positive σ), there is one well-known property that they all share. The **68–95–99.7 rule**, also known as the **3σ rule** or **empirical rule**, states the following:

For a normal distribution with mean μ and standard deviation σ,

- Approximately **68% of the data points lie within 1 standard deviation (1σ) of the mean μ**. See the yellow area in the picture below.
- Approximately **95% of the data points lie within 2 standard deviations (2σ) of the mean μ**. See the pink and yellow areas combined below.
- Approximately **99.7% of the data points lie within 3 standard deviations (3σ) of the mean μ**. See the blue, pink, and yellow areas combined below.

The empirical rule is useful in estimating probabilities for data that is modeled by a normal distribution.

Distribution Plot
Normal with mean 0 and standard deviation 1

68% of data within 1σ of μ

95% of data within 2σ of μ

99.7% of data within 3σ of μ
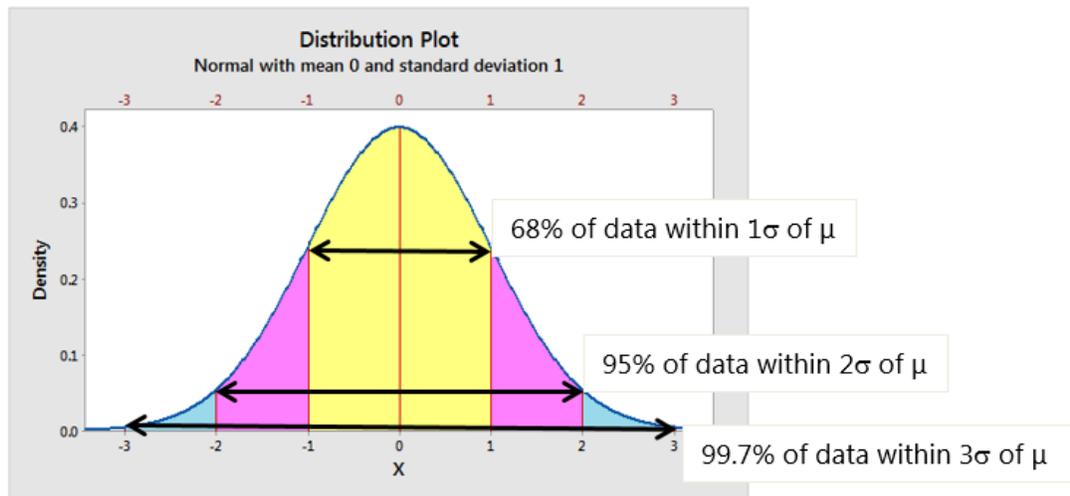
Example 1

Assume IQ scores are normally distributed with mean 100 and standard deviation 10. Determine the probability that a randomly chosen person has an IQ:

(a) Less than 90

(b) Greater than 120

(c) Between 100 and 120

First, let's graph the distribution in Minitab.

Minitab desktop (20 or higher)

1 Choose **Graph > Probability Distribution Plot**.
2 Choose **View Single**, then click **OK**.
3 From **Distribution**, choose **Normal**.
4 In **Mean**, type *100*.
5 In **Standard Deviation**, type *10*.
6 Click **OK**.

Minitab web app

1. Choose **Graph > Probability Distribution Plot**.
2. Choose **View Distribution**, then click **OK**.
3. From **Distribution**, choose **Normal**.
4. In **Mean**, type *100*.
5. In **Standard Deviation**, type *10*.
6. Click **OK**.

Minitab produces the following graph:
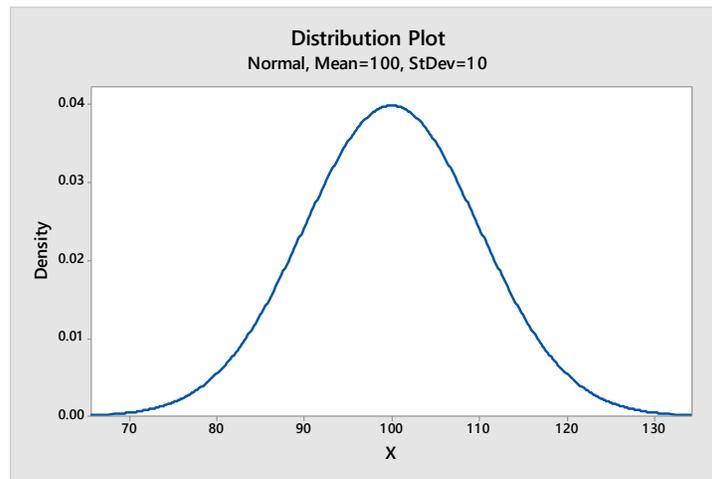
The curve is centered at μ = 100 and the spread is σ = 10. We can find the desired probabilities using the empirical rule.

(a) The probability that a randomly chosen person has an IQ less than 90 is approximately 16%.

- By the empirical rule, approximately 68% of IQ's are within 1σ of the mean μ = 100. Therefore approximately 68% of IQ's are between 90 and 110 with approximately 32% outside 90 and 110.
- By symmetry, approximately 16% of IQ's are less than 90 (and approximately 16% are greater than 110).

We can use Minitab to compute this probability.

Minitab desktop (20 or higher)

1. Choose **Graph > Probability Distribution Plot**.
2. Choose **View Probability**, then click **OK**.
3. From **Distribution**, choose **Normal**.
4. In **Mean**, type *100*.

5   In **Standard Deviation**, type *10.*
6   Click the **Shaded Area** tab. Under **Define Shaded Area By**, choose **X Value**.
7   Click **Left Tail**, since we want the probability of an IQ being less than 90. In **X value**, type *90.*
8   Click **OK**.

Minitab web app

1   Choose **Graph > Probability Distribution Plot**.
2   Choose **View Probability**, then click **OK**.
3   From **Distribution**, choose **Normal**.
4   In **Mean**, type *100.*
5   In **Standard Deviation**, type *10.*
6   Click **Options** and choose **A specified x value**.
7   Click **Left Tail**, since we want the probability of an IQ being less than 90. In **X value**, type *90.*
8   Click **OK**.

Not only does Minitab report the probability (15.87%), but it also shades the portion of the normal distribution plot corresponding to the desired probability.


Distribution Plot
Normal, Mean=100, StDev=10
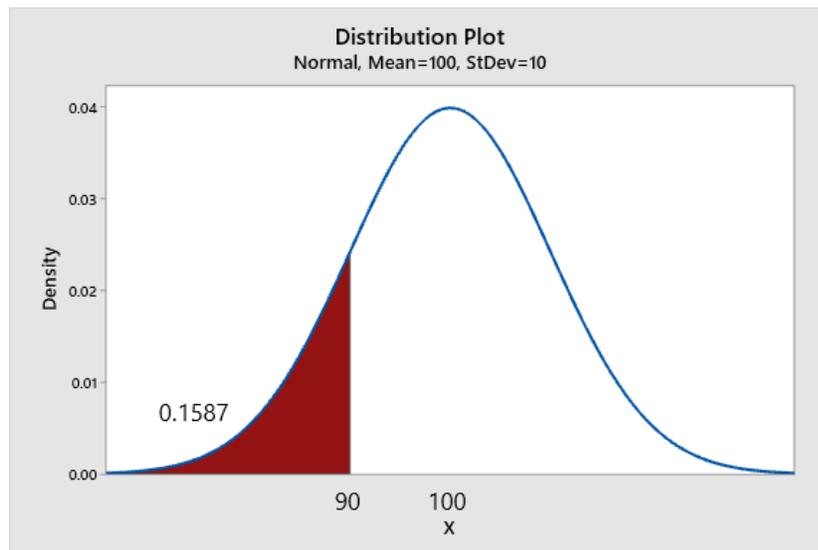
(b) The probability that a randomly chosen person has an IQ greater than 120 is approximately 2.5%.

- By the empirical rule, approximately 95% of IQ's are within 2σ of the mean $\mu = 100$. Therefore approximately 95% of IQ's are between 80 and 120 with approximately 5% outside 80 and 120.

- By symmetry, approximately 2.5% of IQ's are greater than 120 (and approximately 2.5% are less than 80).

We can calculate this probability in Minitab as we did for part **(a)**. The only difference is that we want to view the shaded area corresponding to the **Right Tail** for a specific x value—in this case, our x value is **120**, since we want to determine the probability of an IQ being *greater than 120*. The graph below on the left shows the desired probability (2.3%).
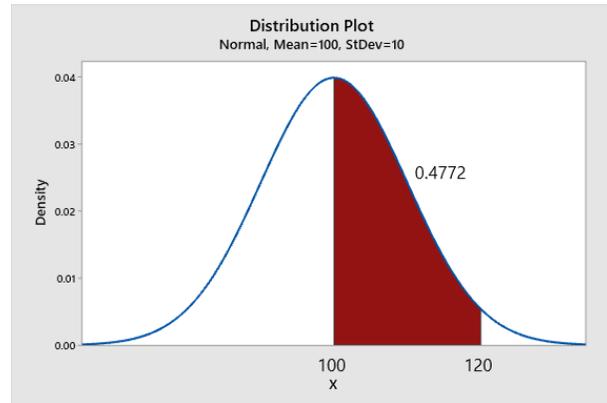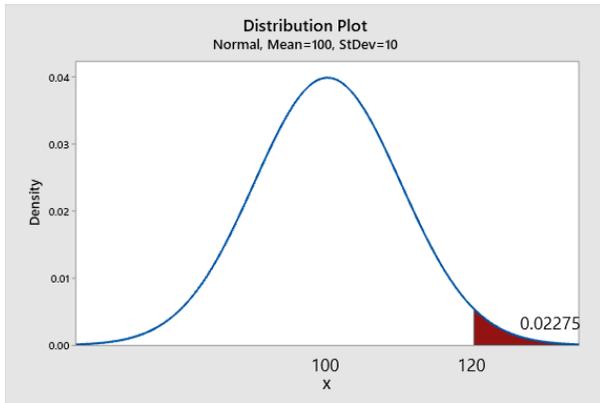
(c) The probability that a randomly chosen person has an IQ between 100 and 120 is...

- By the empirical rule, approximately 95% of IQ's are within 2σ of the mean μ = 100. Therefore approximately 95% of IQ's are between 80 and 120.
- By symmetry, approximately 47.5% of IQ's are between 100 and 120.

We can calculate this probability in Minitab as we did in **(a)** and **(b)**. Now, we need to view the shaded area corresponding to the **Middle** for two x values—in this case, **X value 1** is **100** and **X value 2** is **120**, since we want to determine the probability that a person's IQ is *between 100 and 120*. The graph above on the right shows the desired probability (47.7%).

Computing *z*-Scores

Considering the empirical rule and its application to all normal distributions, it is useful to talk about a value from a normal distribution in terms of the number of standard deviations it is from the mean. That is, we can measure the distance between an *x* value and the center at mean μ in units of size σ. Converting a value to these distance units is called **standardizing**, or **standardization**, where a standardized value is called a **z-score**.

Suppose the data value *x* is from a normal distribution with mean μ and standard deviation σ. To **standardize *x***, we

- Subtract the mean μ from *x*.
- Divide this quantity, *x* – μ, by the standard deviation σ.
- In symbolic terms, a **z-score** is defined as:

$$z = \frac{x - \mu}{\sigma}$$

A *z*-score measures the number of standard deviations between an *x* value and the mean μ. If a *z*-score is positive, then the *x* value is to the right of the mean μ. If a *z*-score is negative, then the *x* value is to the left of the mean μ.

Example 2

Assume IQ scores are normally distributed with mean 100 and standard deviation 10. Determine the *z*-scores for an IQ of 90 and an IQ of 120.

The *z*-score for 90 is:

$$z = \frac{90 - 100}{10} = -1$$

This *z*-score indicates that the value 90 is 1 standard deviation to the left of the mean μ = 100.

The *z*-score for 120 is:

$$z = \frac{120 - 100}{10} = 2$$

This *z*-score indicates that the value 120 is 2 standard deviations to the right of the mean μ = 100.

Normal Distribution with Mean = 100 and Standard Deviation = 10
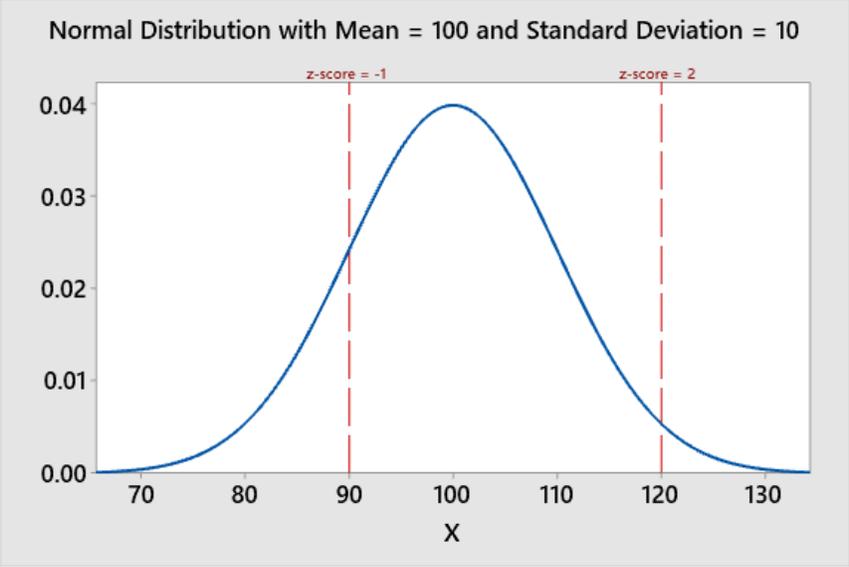
Random Variables

Going forward, we need a way to represent the variables that we have been discussing.

- It is common for statisticians to use capital letters, such as X or Y, to represent random quantities or variables.
- In the first two examples, we could let the random variable X represent a person's IQ.
- We use lowercase letters to represent specific values that the random variable X can assume. In **Example 1**, a specific value that X can attain is $x = 90$.
- Statisticians and textbooks reserve the capital letter Z to represent a **standard normal random variable**. A standard normal random variable is a special normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$.
- When we standardize a normal random variable X with mean μ and standard deviation σ, we obtain a standard normal random variable Z.
- Just as we standardized a specific value x, we can transform any normal random variable X to a standard normal random variable Z.

If X is a normal random variable with mean μ and standard deviation σ, then the random variable:

$$Z = \frac{X - \mu}{\sigma}$$

is a **standard normal random variable** with **mean 0** and **standard deviation 1**.

Since any normal random variable can be converted to a standard normal random variable, a "normal table" appears in statistics textbooks to determine the area under the standard normal curve. This area corresponds to the proportion of data values or the probability of being beyond some z value on the standard normal curve. You can find a normal table at the end of this lesson.

Using the Normal Table to Compute Probabilities

- Again, in order to use this table to determine probabilities associated with a normal distribution, a normal variable must be transformed into the standard normal distribution with $\mu = 0$ and $\sigma = 1$.
- A typical standard normal table, like the one provided at the end of this lesson, provides values for determining the probability of the normal variable Z being **less than or equal to** or strictly less than some designated value z. That is, the standard normal table is used to calculate the cumulative probability $P(Z \leq z)$.

- **Example 3** shows how to compute a probability for a standard normal random variable using a normal table.

Example 3

Assume Z is a standard normal random variable. Determine the following probabilities using the standard normal table provided at the end of this lesson. Shade the area(s) of interest on the graphs.

**(a)** Determining a "less than" probability: $P(Z \leq 1.25)$

$P(Z \leq 1.25)$ is the area to the left of 1.25 on the standard normal curve. This is the type of area, less than a designated value, which is directly given in the normal table.

Look up the value $z = 1.25$ in the table by doing the following:

- The first two digits of the value, **1.2**, are contained on the left hand side of the table.
- The last decimal place, **5**, is shown along the top of the table.
- The corresponding area (probability) is shown in the body of the table. It is approximately 0.89435.

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189

Distribution Plot
Normal, Mean=0, StDev=1

0.8944

To determine P($Z \leq 1.25$) in Minitab as shown in the previous normal curve graph, follow the steps from **Example 1**, reiterated below:

Minitab desktop (20 or higher)

1 Choose **Graph > Probability Distribution Plot**.
2 Choose **View Probability**, then click **OK**.
3 From **Distribution**, choose **Normal**.
4 In **Mean**, type *0*.
5 In **Standard Deviation**, type *1*.
6 Click the **Shaded Area** tab. Under **Define Shaded Area By**, choose **X Value**.
7 Click **Left Tail**, since we want the probability of *Z* being less than 1.25. In **X value** type *1.25*.
8 Click **OK**.

Minitab web app

1 Choose **Graph > Probability Distribution Plot**.
2 Choose **View Probability**, then click **OK**.
3 From **Distribution**, choose **Normal**.
4 In **Mean**, type *0*.
5 In **Standard Deviation**, type *1*.
6 Click **Options** and choose **A specified x value**.
7 Click **Left Tail**, since we want the probability of *Z* being less than 1.25. In **X value** type *1.25*.
8 Click **OK**.

Below is an alternative way to calculate these probabilities.

Minitab desktop (20 or higher)

    1   Choose **Calc > Probability Distributions > Normal**.
    2   Choose **Cumulative Probability**.
    3   In **Mean**, enter *0*. In **Standard Deviation**, enter 1.
    4   Choose **Input constant** and enter *1.25*. Click **OK**.

Minitab web app

    1   Choose **Calc > Probability Distributions > Cumulative Distribution Function**.
    2   From **Form of input,** select **A single value**.
    3   In **Value**, enter *1.25*.
    4   In **Mean**, enter *0*. In **Standard Deviation**, enter 1.
    5   Under **Output**, select **Display a table of cumulative probabilities**.
    6   Click **OK**.

(b) Determining a "greater than" probability: P(Z > 2.07)

P(Z > 2.07) is the area to the right of 2.07 on the standard normal curve.

Look up the value z = 2.07 in the table by doing the following:

- The first two digits of the value, *2.0*, are contained on the left hand side of the table.
- The last decimal place, *7*, is shown along the top of the table.
- The area (probability) in the body of the table is the area to the **left** of 2.07. Since the total area under the curve is 1, the area to the right of 2.07 is approximately 1 − 0.98077 = 0.01923.

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574

To determine P(Z > 2.07) in, do the following:

Minitab desktop (20 or higher)

1. Choose **Graph > Probability Distribution Plot**.
2. Choose **View Probability**, then click **OK**.
3. From **Distribution**, choose **Normal**.
4. In **Mean**, type *0*.
5. In **Standard Deviation**, type *1*.
6. Click the **Shaded Area** tab. Under **Define Shaded Area By**, choose **X Value**.
7. Click **Right Tail**, since we want the probability of *Z* being greater than 2.07. In **X value** type *2.07*.
8. Click **OK**.

Minitab web app

1. Choose **Graph > Probability Distribution Plot**.
2. Choose **View Probability**, then click **OK**.
3. From **Distribution**, choose **Normal**.
4. In **Mean**, type *0*.

5   In **Standard Deviation**, type 1.
6   Click **Options** and choose **A specified z value**.
7   Click **Right Tail**, since we want the probability of *Z* being greater than 2.07. In **X value** type *2.07*.
8   Click **OK**.

(c) Determining a "between" probability: P(0.56 < *Z* < 1.78)

P(0.56 < *Z* < 1.78) is the area between 0.56 and 1.78 on the standard normal curve. We can write this probability as the difference of two "less than" probabilities: P(0.56 < *Z* < 1.78) = P(*Z* < 1.78) – P(*Z* < 0.56). We need to look up the values *z* = 1.78 and *z* = 0.56 in the standard normal table.

Look up the value *z* = 1.78 in the table by doing the following:

- The first two digits of the value, **1.7**, are contained on the left hand side of the table.
- The last decimal place, **8**, is shown along the top of the table.
- The area (probability) in the body of the table is the area to the left of 1.78, which is approximately 0.96246.

Look up the value *z* = 0.56 in the table by doing the following:

- The first two digits of the value, **0.5**, are contained on the left hand side of the table.
- The last decimal place, **6**, is shown along the top of the table.
- The area (probability) in the body of the table is the area to the left of 0.56, which is approximately 0.71226.

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327

Subtract P(*Z* < 0.56) from P(*Z* < 1.78), which yields 0.96246 – 0.71226 = 0.2502.

We can check this in Minitab as we did for the previous two parts. Instead of selecting Left Tail or Right Tail, we need to select **Middle** in the Shaded Area menu and type **0.56** for **X value 1** and **1.78** for **X value 2**.

Minitab returns the following graph:
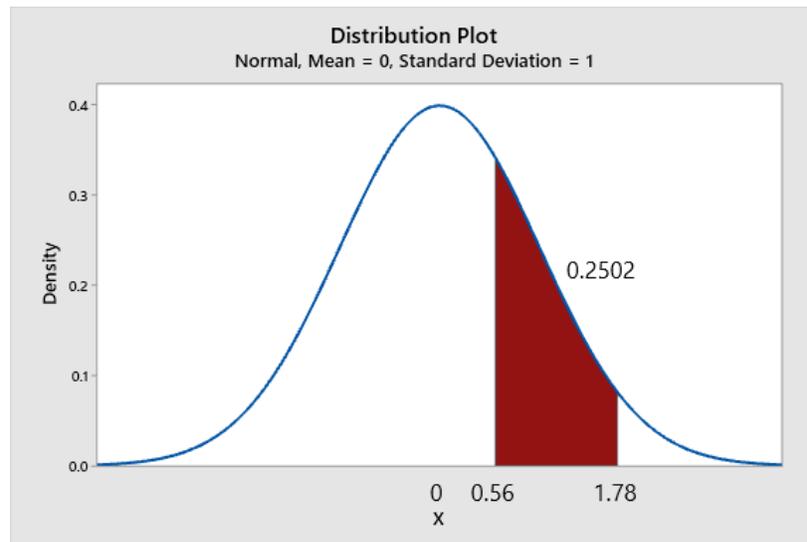
We can now use the standard normal table to compute any probability associated with a normal distribution by using z-scores. The next three examples illustrate this.

Example 4

The time it takes a cell to divide (called mitosis) is normally distributed with an average of one hour and a standard deviation of 5 minutes. What proportion of dividing times will take less than 65 minutes?

If we let X represent the time for a cell to divide, then X has a normal distribution with mean μ = 60 minutes and standard deviation σ = 5 minutes. The z-score for the time 65 minutes is:

$$z = \frac{65 - 60}{5} = 1$$

The probability that the cell division time is less than 65 minutes can be found using the normal table.

$$P(X < 65) = P(Z < 1) \cong 0.84134$$

Computing this probability in Minitab, we get the same answer.

Distribution Plot
Normal, Mean = 60, Standard Deviation = 5

0.8413
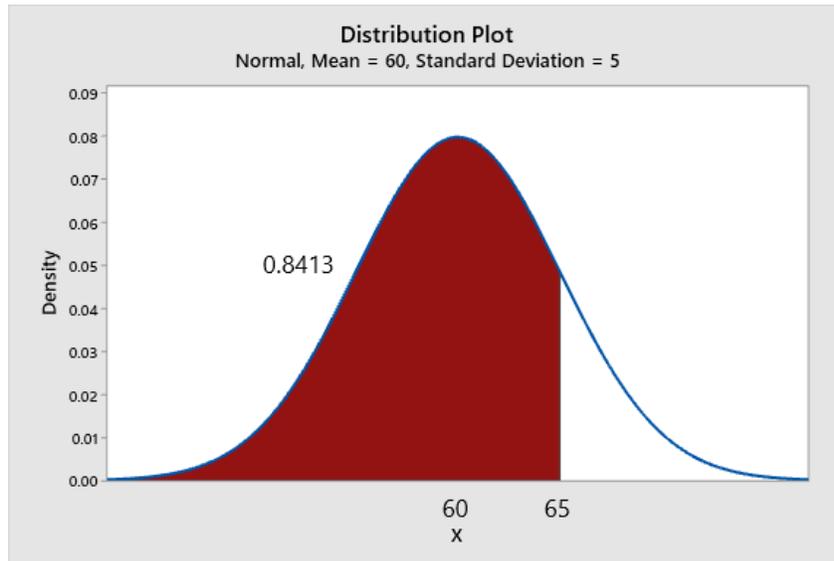
Example 5

Our dog Bella has a vertical jumping height that is normally distributed with mean 3 feet and standard deviation 0.8 feet. We're considering training her to become an agility course dog. What's the probability that Bella can jump higher than the 2.4-foot bench in our backyard?

Let X represent Bella's jumping height. Then X has a normal distribution with mean μ = 3 feet and standard deviation σ = 0.8 feet. The z-score for the height 2.4 feet is:

$$z = \frac{2.4 - 3}{0.8} = -0.75$$

The probability that her jumping height is greater than 2.4 feet can be found by using the normal table.

$$P(X > 2.4) = P(Z > -0.75) \cong 1 - 0.22663 = \mathbf{0.77337}$$

Computing this probability in Minitab, we get the same answer.

Distribution Plot
Normal, Mean = 3, Standard Deviation = 0.8

0.7734
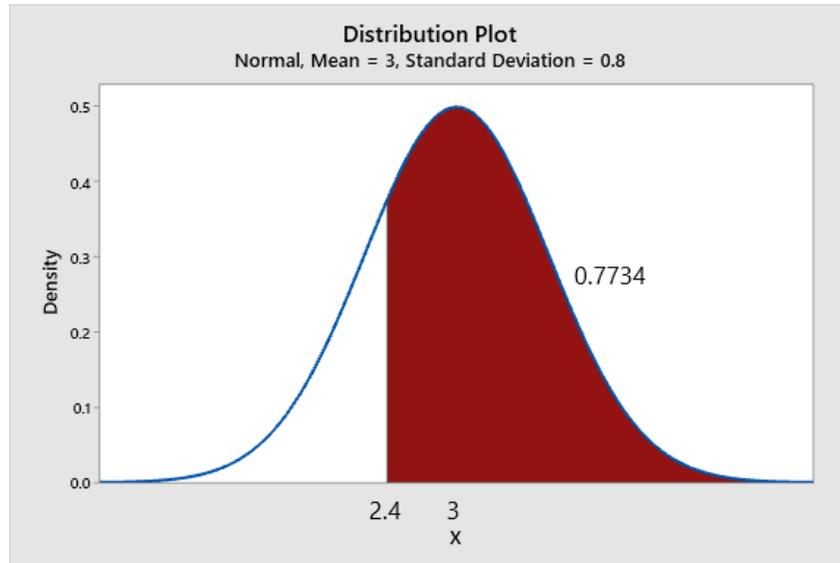
Example 6

The time that it takes a driver to react to the brake lights on a decelerating vehicle is critical in avoiding rear-end collisions. An article in the journal *Cars* suggests that reaction time for an in-traffic response to a brake signal from standard brake lights can be modeled by a normal distribution with mean 1.25 seconds and standard deviation 0.46 seconds. What proportion of reaction times will be between 1.00 and 1.75 seconds?

If we let X represent the reaction time, then X has a normal distribution with mean μ = 1.25 seconds and standard deviation σ = 0.46 seconds. The z-scores for the times 1.00 second and 1.75 seconds are:

$$z = \frac{1.00 - 1.25}{0.46} \cong -0.54 \text{ and } z = \frac{1.75 - 1.25}{0.46} \cong 1.09$$

The probability that the reaction time is between 1 and 1.75 seconds can be found using the normal table.

$$P(1 < X < 1.75) = P(X < 1.75) - P(X < 1) = P(Z < 1.09) - P(Z < -0.54) \cong 0.86214 - 0.29460$$
$$= \mathbf{0.56754}$$

Computing this probability in Minitab, we get a slightly more accurate answer than provided by the normal table.

Distribution Plot
Normal, Mean = 1.25, Standard Deviation = 0.46
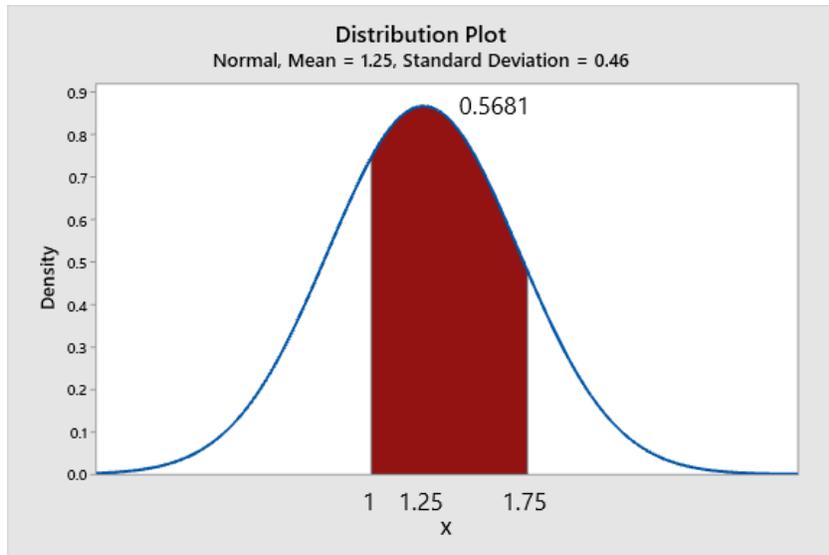
Inverse Normal Calculations

Examples 4 – 6 use the normal distribution to find the proportion (probability) of observations for a given event, such as "reaction time to brake lights between 1 and 1.75 seconds."

- We may want to go "backwards" and compute the observed value x corresponding to a certain proportion, such as the proportion of reaction times to brake lights that are more than 2 seconds.
- We'll need to work in the opposite direction with the normal table:
  - Use the proportion, which is inside the normal table, to find its corresponding z-score.
  - Transform the z-score back to the original x scale (when applicable).

Example 7

What z-score corresponds to the proportion 0.1 in the left tail of a standard normal distribution?

Drawing a picture of the normal distribution with the corresponding proportion is a good way to start.

Distribution Plot
Normal, Mean = 0, Standard Deviation = 1
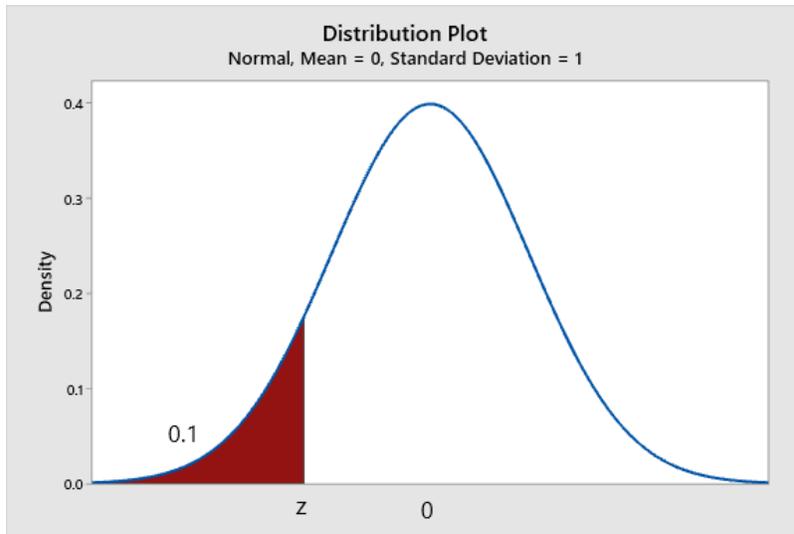
To determine the *z*-score using the normal table, look in the body of the normal table for the value closest to 0.1. It is **z = -1.28**.

Example 8

What *z*-score corresponds to the proportion 0.15 in the right tail of a standard normal distribution?
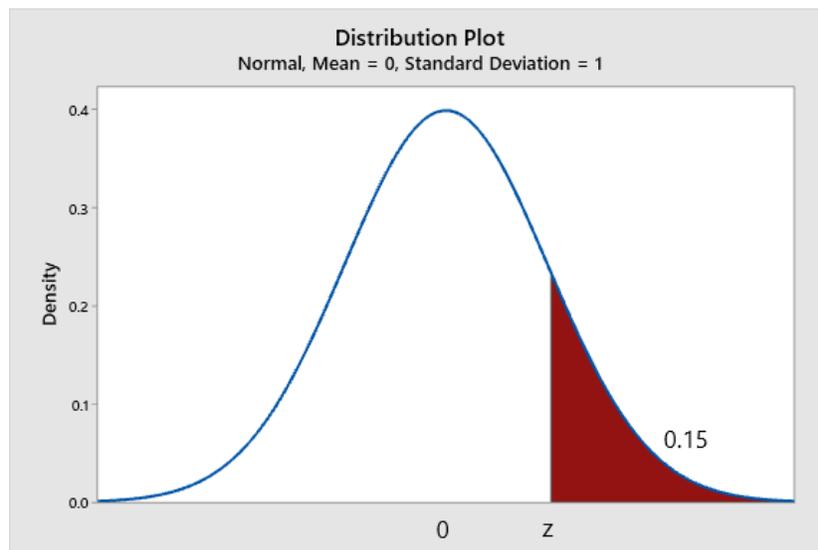
Distribution Plot
Normal, Mean = 0, Standard Deviation = 1

Since we are determining a right tail proportion, determine the *z*-score in the normal table corresponding to 1 − 0.15 = 0.85. Looking in the body of the normal table for the value closest to 0.85 gives us **z = 1.04**.

Let's now determine the *x* value for any normal distribution with mean μ and standard deviation σ. In addition to using the normal table as we did in **Examples 7** and **8**, we have to add a step and transform the z-score back to the original x scale.

Example 9

Mensa (from the Latin word "mind") is an international society devoted to intellectual pursuits. Any person who has an IQ in the upper 2% of the general population is eligible to join. Assume that IQs are normally distributed with μ = 100 and σ = 15. What is the lowest IQ that will qualify a person for Mensa?

Let *X* represent a person's IQ score. We are trying to determine the *x* value shown in the graph below.

Distribution Plot
Normal, Mean = 100, Standard Deviation = 15
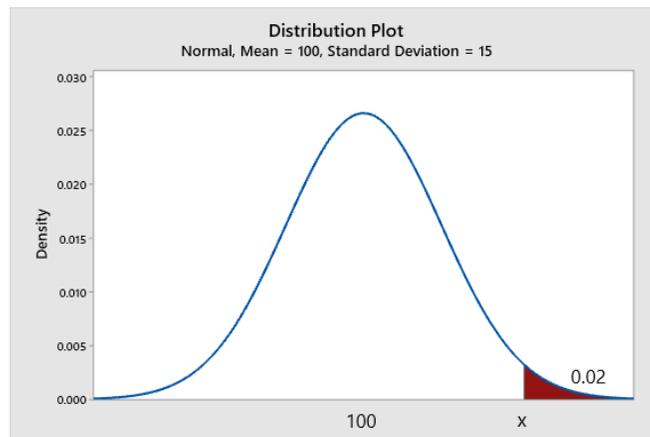
- Since we are using a right-tail proportion, determine the *z*-score in the normal table corresponding to 1 – 0.02 = 0.98. Looking in the body of the normal table for the value closest to 0.98 gives us **z = 2.06**.
- Now we need to "unstandardize" the *z*-score back to the original *x* (IQ) scale. We know that *x* must satisfy the following equation:

$$\frac{x - 100}{15} = 2.06$$

- Solving this equation for *x* gives us an IQ score of **130.9.**

$$x - 100 = 2.06 \cdot 15$$
$$x = 100 + 2.06 \cdot 15$$
$$x = 130.9$$

To determine this value in Minitab, do the following:

Minitab desktop (20 or higher)

1. Choose **Graph > Probability Distribution Plot**.
2. Choose **View Probability**, then click **OK**.
3. From **Distribution**, choose **Normal**.
4. In **Mean**, type *100*.
5. In **Standard Deviation**, type *15*.
6. Click the **Shaded Area** tab. Under **Define Shaded Area By**, choose **Probability**.
7. Click **Right Tail**, since we want the *x* value corresponding to 0.02 in right tail. In **Probability**, type *0.02*.
8. Click **OK**.

Minitab web app

1. Choose **Graph > Probability Distribution Plot**.
2. Choose **View Probability**, then click **OK**.
3. From **Distribution**, choose **Normal**.
4. In **Mean**, type *100*.
5. In **Standard Deviation**, type *15*.
6. Click **Options** and choose **A specified probability**.
7. Click **Right Tail**, since we want the *x* value corresponding to 0.02 in right tail. In **Probability**, type *0.02*.
8. Click **OK**.

Minitab determines a slightly more accurate value of *x* = **130.8**.
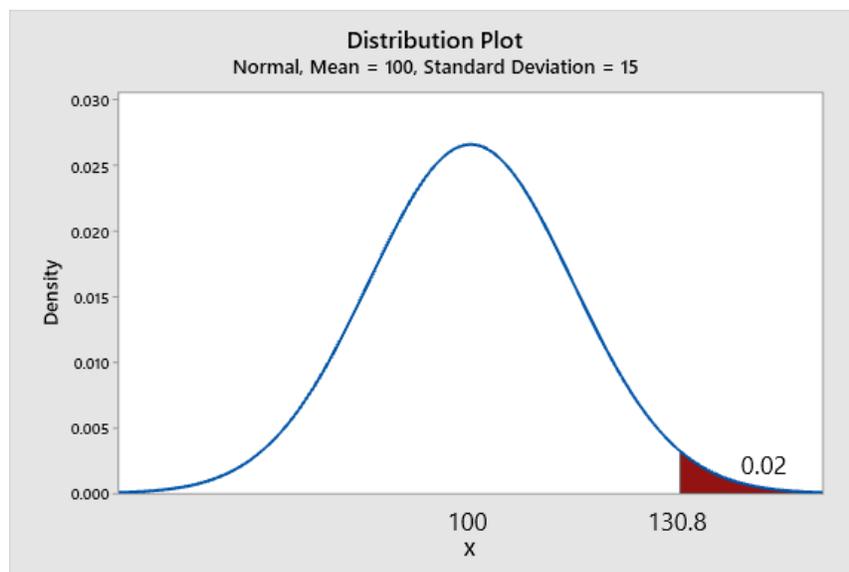
Example 10

Suppose that the lifetime of a given type of light bulb can be modeled by a normal distribution with mean 1000 hours and standard deviation 100 hours. Determine the lifetime such that 90% of light bulbs surpass it.

Let X represent the lifetimes of these light bulbs.

- Since we are determining a right tail proportion, determine the z-score in the normal table corresponding to 1 – 0.90 = 0.10. Looking in the body of the normal table for the value closest to 0.10 gives us **z = -1.28**.
- Now we need to "unstandardize" the z-score back to the original x (lifetime of bulb) scale. We know that x must satisfy the following equation:

$$\frac{x - 1000}{100} = -1.28$$

- Solving this equation for x gives us a lifetime of **872 hours.**

Using Minitab and the instructions for **Example 9**, we obtain x = 871.8 hours.

Distribution Plot
Normal, Mean = 1000, Standard Deviation = 100
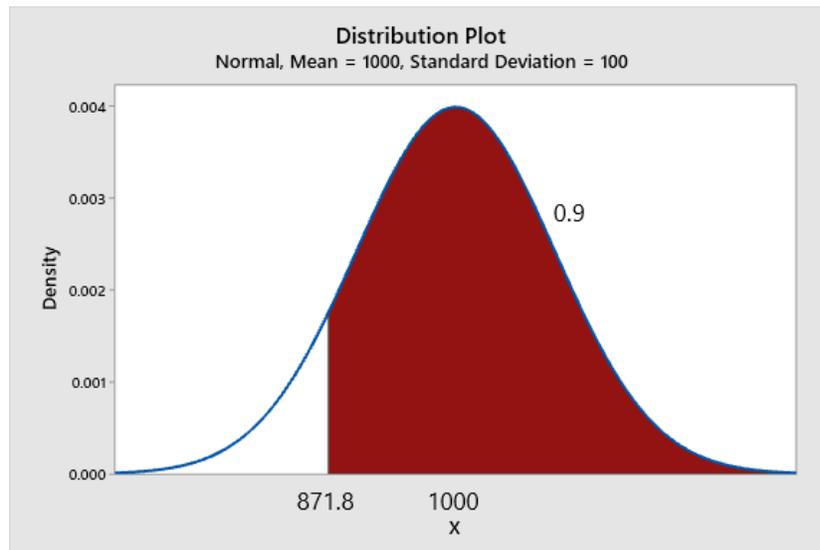
Using the Standard Normal Distribution Table

The graph below depicts how to interpret the standard normal distribution tables provided on the following pages.
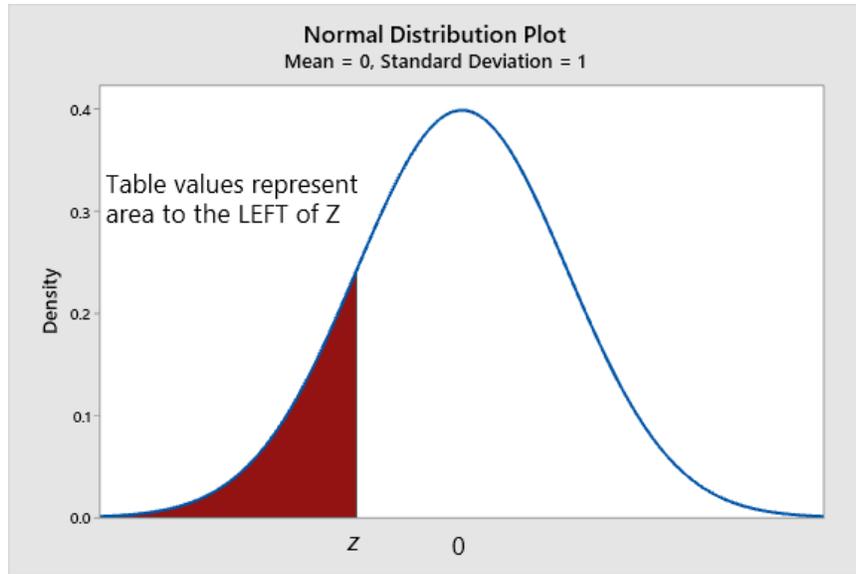
Normal Distribution Plot
Mean = 0, Standard Deviation = 1

Table values represent area to the LEFT of Z

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997